ON STRONGLY AND WEAKLY DEFINED BOOLEAN TERMS

BY

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ABSTRACT

A problem of Gaifman about strongly and weakly defined Boolean terms is solved by finding a Boolean algebra $\mathscr{F}$ with a complete subalgebra $\mathscr{E}$ such that some element of $\mathscr{F}$ not in $\mathscr{E}$ can be obtained from elements of $\mathscr{E}$ by meets and joins in the normal completion of $\mathscr{F}$.

Introduction

In this paper we solve a problem posed by Gaifman in his paper [1, §0]. Roughly speaking, the problem is as follows. Let $\mathscr{B}$ be a Boolean algebra, $\mathscr{C}$ its normal completion, and I an assignment of values in $\mathscr{B}$ to some variables. Let ψ be a Boolean term on these variables (i.e., constructed from them by the unary operation $\neg$ and the infinitary operations $\bigwedge$, $\bigvee$), and suppose that the value of ψ as computed in $\mathscr{C}$ under the assignment I (interpreting $\neg, \bigwedge, \bigvee$ as complement, meet and join in $\mathscr{C}$) turns out to be an element of $\mathscr{B}$. Does there always exist a Boolean term ϕ such that (1) ϕ is equivalent to ψ, and (2) ϕ is defined in $(\mathscr{B}, I)$?

Part (1) means that ϕ and ψ get the same value in all assignments into complete Boolean algebras. Part (2) means that the value of ϕ under I can be computed directly in $\mathscr{B}$, so that all meets and joins needed in the process exist in $\mathscr{B}$.

In §2, we give an example showing that such a ϕ need not *always* exist. But in §3–4, we prove that the answer is affirmative if we restrict ourselves to pairs $(\mathscr{B}, I)$ satisfying a simple regularity condition. A more precise statement of the problem and results is given in §1, where the basic terminology and notations concerning Boolean terms are explained.

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Our set-theoretical notations are standard: $\sim$, $\cap$, $\cup$, $\bigcap$, $\bigcup$ are the usual operations on sets. A field of subsets of X is, by definition, a set $\mathscr{F}$ of subsets of X such that $X \in \mathscr{F}$ and $A, B \in \mathscr{F} \Rightarrow X \sim A$, $A \cap B$, $A \cup B \in \mathscr{F}$. For us, $2 = \{0,1\}$ and $\omega = \{0,1,2,3,\cdots\}$.

The symbols $\neg$, $\wedge$, $\vee$, $\bigwedge$, $\bigvee$, are used either to denote the operations of a Boolean algebra (and then we often write $\neg^{\mathscr{B}}$, $\bigwedge^{\mathscr{B}}$, $\bigvee^{\mathscr{B}}$), or as symbolic operations (connectives) on Boolean terms. $\forall$, $\exists$ are sometimes used as abbreviations of English phrases (for all, there exists).

The main results of the paper, which is a part of the author's Ph.D. thesis prepared at the Hebrew University of Jerusalem under the direction of Prof. H. Gaifman, were announced in [3]. I wish to thank Prof. Gaifman for his interest and advice throughout the work.

The paper can be understood by any reader having an elementary knowledge of Boolean algebras and topology.

1. Preliminaries and formulation of the problem and results

Our terminology and notation differ slightly from Gaifman's. Let D be a fixed set (in [1] D is taken as an ordinal δ). Consider variables p_i, $i \in D$, which will assume values in arbitrary Boolean algebras. The Boolean terms (B.t's) over D are defined inductively by:

p_i is a B.t. for $i \in D$;

if ϕ is a B.t. then $\neg \phi$ is a B.t.;

if X is a set of B.t's, then $\bigwedge X$ and $\bigvee X$ are B.t's.

One defines $\phi \wedge \psi = \bigwedge\{\phi,\psi\}$, $\phi \vee \psi = \bigvee\{\phi,\psi\}$, $(\phi \rightarrow \psi) = \neg \phi \vee \psi$, $(\phi \leftrightarrow \psi) = (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

A valuation over D is a pair $(\mathscr{B}, I)$ consisting of a Boolean algebra (B.a.) $\mathscr{B}$ and a function $I: D \rightarrow \mathscr{B}$. D will usually be fixed and all B.t's and valuations are understood to be over D. One is tempted to define the value $\|\phi\| = \|\phi\|_{\mathscr{B},I}$ of a B.t. in a valuation by the following equations:

$$p_i = I(i) \text{ for } i \in D;$$

$$\|\neg\psi\| = \neg^{\mathscr{B}}\|\psi\|;$$

$$\|\bigwedge X\| = \bigwedge^{\mathscr{B}}_{\psi \in X} \|\psi\| \text{ and dually for } \bigvee X.$$

If $\mathscr{B}$ is complete, these equations determine a value in $\mathscr{B}$ for each B.t. In the general case we may agree that ϕ is not defined in $(\mathscr{B}, I)$ when the computation of $\| \phi \|_{\mathscr{B}.I}$ by means of the above equations fails at a point because some meet or join does not exist in $\mathscr{B}$. If ϕ *is* defined in $(\mathscr{B},I)$ then $\| \phi \|_{\mathscr{B}\ I}$ is a certain element of $\mathscr{B}$, and we say for emphasis that ϕ is strongly defined in $(\mathscr{B},I)$. An exact definition by induction on ϕ is left to the reader.

1.1. LEMMA. *Let $\mathscr{B}_1,\mathscr{B}_2$ be B.a's, $I: D \to \mathscr{B}_1$ and h a complete homomorphism of $\mathscr{B}_1$ into $\mathscr{B}_2$. If the B.t. ϕ is strongly defined in $(\mathscr{B}_1,I)$, then ϕ is strongly defined in $(\mathscr{B}_2, h \circ I)$, and $h(\| \phi \|_{\mathscr{B}_1,I}) = \| \phi \|_{\mathscr{B}_2,h \circ I}$.*

PROOF. Obvious by induction on ϕ.

Recall that $\mathscr{B}_1$ is called a regular subalgebra of $\mathscr{B}_2$ when it is a subalgebra and the inclusion embedding of $\mathscr{B}_1$ in $\mathscr{B}_2$ is complete. A subalgebra $\mathscr{B}_1$ of $\mathscr{B}_2$ is called dense when every $b \in \mathscr{B}_2$ is a join in $\mathscr{B}_2$ of members of $\mathscr{B}_1$. Every dense subalgebra is regular. By a normal completion of $\mathscr{B}$ we mean a complete B.a. $\mathscr{C}$ of which $\mathscr{B}$ is a dense subalgebra. A well-known theorem (see [2, §35]) states that every B.a. $\mathscr{B}$ has a normal completion, and if $\mathscr{C}_1, \mathscr{C}_2$ are normal completions of $\mathscr{B}$ then an isomorphism of $\mathscr{C}_1$ and $\mathscr{C}_2$ exists which acts as the identity on $\mathscr{B}$.

We shall use the following known property of the normal completion:

1.2. *Let $\mathscr{B}$ be a B.a., $\mathscr{C}$ a normal completion of $\mathscr{B}$ and $\mathscr{C}'$ a complete B.a. Every complete homomorphism $j: \mathscr{B} \to \mathscr{C}'$ has a unique extension to a complete homomorphism $J: \mathscr{C} \to \mathscr{C}'$. If j is one-one, so is J.*

PROOF. For any $c \in \mathscr{C}$ let $R_c = \{x \in \mathscr{B} \mid x \leqq c\}$. Define J by $J(c) = \bigvee^{\mathscr{C}'} j'' R_c$ for any $c \in \mathscr{C}$. Using the facts that $x \in R_c,\ y \in R_{\neg c} \Rightarrow (x \wedge y = 0$ in $\mathscr{B})$ and that $\bigvee^{\mathscr{B}}(R_c \cup R_{\neg c}) = 1$ for all $c \in \mathscr{C}$, one easily concludes that J preserves complements. Now suppose that $c = \bigwedge^{\mathscr{C}} A (A \subseteq \mathscr{C},\ c \in \mathscr{C})$. Then in $\mathscr{C}'$, $J(c) = \bigvee j'' R_c$ while $\bigwedge_{a \in A} J(a) = \bigwedge_{a \in A} \bigvee j'' R_a$. But $a \in A \Rightarrow c \leqq a \Rightarrow R_c \subseteq R_a \Rightarrow J(c) \leqq J(a)$ so clearly $J(c) \leqq \bigwedge_{a \in A} J(a)$. To prove equality it suffices to prove that in $\mathscr{C}'$, $J(c) \vee \neg \bigwedge_{a \in A} J(a) = 1$, i.e., $J(c) \vee \bigvee_{a \in A} J(\neg a) = 1$; equivalently, $\bigvee j''(R_c \cup \bigcup_{a \in A} R_{\neg a}) = 1$. Since j is complete it is enough to show that $\bigvee^{\mathscr{B}}(R_c \cup \bigcup_{a \in A} R_{\neg a}) = 1$, but this is true because $\mathscr{B}$ is dense in $\mathscr{C}$ and $c \vee \bigvee_{a \in A} \neg a = 1$ in $\mathscr{C}$. Thus J preserves arbitrary meets, and hence is complete. The uniqueness of J is obvious.

If j is one-one so is J because if $c \in \mathscr{C}$, $c \neq 0$, then there is some $x \in R_c$ such that $x \neq 0$, and so $J(c) \geqq j(x) > 0$. This completes the proof.

Let us say that a B.t. ϕ is weakly defined in the valuation $(\mathscr{B},I)$ when $\| \phi \|_{\mathscr{C}.I} \in \mathscr{B}$ where $\mathscr{C}$ is any normal completion of $\mathscr{B}$ (the choice of $\mathscr{C}$ does not matter because $\mathscr{C}$ is essentially unique).

Note that by 1.1, if ϕ is strongly defined in $(\mathscr{B},I)$ then $\|\phi\|_{\mathscr{B},I} = \|\phi\|_{\mathscr{C},I}$ ($\mathscr{C}$ being the normal completion of $\mathscr{B}$) and so ϕ is weakly defined. When D is finite every B.t. is strongly, hence weakly, defined in every valuation over D. But when D is infinite it is easy to find valuations in which some B.t's are weakly but not strongly defined.

Let ϕ, ψ be B.t's. We write $\phi \equiv \psi$ and say that ϕ, ψ are equivalent when $\|\phi\| = \|\psi\|$ in every valuation in which both are (strongly) defined; equivalently, $\|\phi\| = \|\psi\|$ in every valuation $(\mathscr{B},I)$ such that $\mathscr{B}$ is complete. When $\phi \equiv \bigwedge \varnothing$ (i.e., $\|\phi\| = 1$ always) we write $\vdash \phi$. More generally, let $\Gamma \cup \{\phi\}$ be a set of B.t's. We write $\Gamma \vdash \phi$ (read: ϕ is a (Boolean) consequence of Γ) when $\vdash \bigwedge \Gamma \rightarrow \phi$; equivalently, when $\|\phi\| = 1$ in every valuation $(\mathscr{B},I)$ such that $\mathscr{B}$ is complete and $\|\psi\|_{\mathscr{B},I} = 1$ for all $\psi \in \Gamma$. We shall use only some obvious properties of the relation $\vdash$.

Given a subset A of a B.a. $\mathscr{B}$ we let $[A]_{\mathscr{B}}^{<\infty}$ be the smallest $C \supseteq A$ such that C is the underlying set of a $<\infty$-subalgebra (also called complete subalgebra) of $\mathscr{B}$; i.e., such that C is closed under $\neg^{\mathscr{B}}$ and under all meets and joins existing in $\mathscr{B}$. We shall usually identify subalgebras with their underlying sets when the common superalgebra is fixed. Note that a $<\infty$-subalgebra $\mathscr{B}$ of a complete B.a. $\mathscr{C}$ is a regular subalgebra of $\mathscr{C}$, and is complete as a B.a. in itself.

This completes the general preliminaries. The problem posed by Gaifman was whether (*) below is true for all valuations $(\mathscr{B},I)$.

(*) For every B.t. ψ weakly defined in $(\mathscr{B},I)$ there is a B.t. ϕ such that $\phi \equiv \psi$ and ϕ is strongly defined in $(\mathscr{B},I)$.

The question depends on the set D (Gaifman's δ) over which B.t's are considered, and the answer is obviously affirmative for finite D.

Our first main result is that when D is infinite (*) is not always true.

We shall prove this for $D = \omega \times \omega$ (so that the variables are p_{mn}; $m,n < \omega$), but since every infinite set has a countably infinite subset it is not hard to conclude that (*) is sometimes fales for every infinite D. Specifically we shall prove:

1.3. THEOREM. *There is a valuation $(\mathscr{B},I)$ over $\omega \times \omega$ in which the B.t. $\bigvee_n \bigwedge_m p_{mn}$ is weakly defined, but no B.t. equivalent to it is strongly defined.*

Our second main result is that (*) is true when the valuation $(\mathscr{B},I)$ is regular: A valuation $(\mathscr{B},I)$ is called reduced when $\mathscr{B}$ is generated in the $<\infty$-sense by range (I), i.e., $\mathscr{B} = [\text{range (I)}]_{\mathscr{B}}^{<\infty}$. $(\mathscr{B},I)$ is called regular when $[\text{range (I)}]_{\mathscr{B}}^{<\infty}$ is a regular subalgebra of $\mathscr{B}$. Clearly every reduced valuation is regular.

1.4. THEOREM. *If $(\mathscr{B},I)$ is a regular valuation then it satisfies* (*).

Theorems 1.3 and 1.4 are the main results but we shall prove Theorem 1.3 via the following assertion which is of interest in itself.

1.5. *There exists a B.a. $\mathscr{F}$ with a $<\infty$-subalgebra $\mathscr{E}$, elements P_{mn} $(m,n<\omega)$ of $\mathscr{E}$ and an element Q of $\mathscr{F}$ such that $Q\notin\mathscr{E}$ but the equation $Q=\bigvee_n^{\mathscr{C}}\bigwedge_m P_{mn}^{\mathscr{C}}$ holds in the normal completion $\mathscr{C}$ of $\mathscr{F}$.*

(The proof will give $\mathscr{F}$ as a field of sets and $\mathscr{E}$ as a subfield.)

To get 1.3 from this, define the valuation $(\mathscr{B},I)$ as follows: $\mathscr{B}=\mathscr{F}$, and $I:\omega\times\omega\to\mathscr{F}$ is defined by $I(m,n)=P_{mn}$. Let $\mathscr{C}$ be the normal completion of $\mathscr{B}$. Then by 1.5, the B.t. $\psi=\bigvee_n\bigwedge_m p_{mn}$ has the value Q in $(\mathscr{C},I)$ and so is weakly defined in $(\mathscr{B},I)$ since $Q\in\mathscr{B}$. But let ϕ be any B.t. strongly defined in $(\mathscr{B},I)$. Then clearly $\|\phi\|_{\mathscr{B}I}\in\mathscr{E}$ since $\mathscr{E}\supseteq[\text{range } (I)]_{\mathscr{B}}^{<\infty}$ and so $\|\phi\|_{B,I}=\|\phi\|_{\mathscr{C}I}\neq Q$; hence, $\phi\not\equiv\psi$, proving Theorem 1.3.

In §2, we prove 1.5, §3 contains the proof of Theorem 1.4 for reduced valuations, and §4 proves Theorem 1.4 in the general case of regular valuations.

2. Example of a field of sets

We shall consider fields of subsets of the space ${}^{\omega}2=\{x\mid x:\omega\to 2\}$. A basic set is one of the form $B=\{x\in{}^{\omega}2\mid(\forall i<n)x(i)=\delta_i\}$, where $\langle\delta_i\mid i<n\rangle$ is any finite sequence of zeros and ones. An elementary set is a finite union of basic sets. The collection $\mathscr{E}$ of all elementary sets is a field (of subsets of ${}^{\omega}2$). An open set is a (countable) union of basic sets. This makes ${}^{\omega}2$ a topological space (the Cantor space), in which the closed sets are the intersections of sequences of elementary sets, and $\mathrm{cl}(A)=\bigcap\{E\mid E\in\mathscr{E},E\supseteq A\}$ ("cl" is the closure operation). Recall that a closed set is nowhere-dense iff it has no interior points, and that a perfect set is a closed nonempty set having no isolated point.

To prove 1.5 we shall make use of sets $Q,Q',Q_n(n<\omega)$, $R_{nk}(n,k<\omega)$ (all subsets of ${}^{\omega}2$) satisfying the following.

2.1. (1) $Q=\bigcup_n Q_n$ and $Q'={}^{\omega}2\sim Q$;

(2) for each basic set B, $Q\cap B$ and $Q'\cap B$ are uncountable;

(3) the sets Q_n are perfect, nowhere dense and pairwise-disjoint;

(4) for each fixed n, the sets R_{nk} are countable, pairwise-disjoint and satisfy $\mathrm{cl}(R_{nk})=Q_n$.

We begin by showing the existence of sets satisfying 2.1. It is not hard to see, using Baire's category theorem, that if Q is any dense set which is the union of a sequence of perfect nowhere-dense sets, then $Q', Q_n, R_{nk} (n, k < \omega)$ can be found so that 2.1 holds. But the quickest way is to give a particular example. Let $Q = \{x \in {}^{\omega}2 \mid \exists n (\forall m \geqq n)\ (x(2m) = 1)\}$, $Q' = {}^{\omega}2 \sim Q$, $Q_0 = \{x \in {}^{\omega}2 \mid \forall m(x(2m) = 1)\}$ and for $n > 0$,

$$Q_n = \{x \in {}^{\omega}2 \mid (\forall m \geqq n)\ (x(2m) = 1) \text{ and } x(2n-2) = 0\}.$$

Then (1)–(3) of 2.1 are easily verified. Next let:

$$R_{nk} = \{x \in Q_n \mid \text{ the sequence } (x(1), x(3), x(5), x(7), \cdots)$$

has a tail of the form $0 \cdots 010 \cdots 010 \cdots 01 \cdots$ where in each block 0 occurs (consecutively) k times$\}$.

Then 2.1(4) holds too.

From now on we consider any fixed sets Q, Q' etc., so that 2.1 holds. Since the sets Q_n are closed, each has the form $Q_n = \bigcap_m P_{mn}$ where $P_{mn} \in \mathscr{E}$ for all m, n. Thus $Q = \bigcup_n \bigcap_m P_{mn}$.

Let $\mathscr{F}$ be the smallest field of subsets of ${}^{\omega}2$ that contains all elementary sets, Q and R_{nk} for all n, k. In order to prove the assertions of 1.5 (of course, $\mathscr{E}$ and $\mathscr{F}$ are regarded as B.a's with the set operations $\sim$, $\cap$, $\cup$ so that $\leqq$ coincides with set-inclusion) we need only prove the following (note that $Q \notin \mathscr{E}$; this is clear from 2.1).

2.2. $\mathscr{E}$ is a $< \infty$-subalgebra of $\mathscr{F}$.

2.3. $Q = \bigvee_n^{\mathscr{C}} \bigwedge_m^{\mathscr{C}} P_{mn}$ where $\mathscr{C}$ is the normal completion of $\mathscr{F}$.

We begin with the proof of 2.2. Let $\mathscr{R}$ be the ideal in $\mathscr{F}$ generated by $\{R_{nk} \mid n, k < \omega\}$, and for $A, B \in \mathscr{F}$ write $A \approx B$ for: A is congruent to B modulo $\mathscr{R}$, i.e., the symmetric difference $A \triangle B$ can be covered by a finite number of R_{nk}'s (and hence is countable).

Lemma 1. *For each $A \in \mathscr{F}$ there are $E, E' \in \mathscr{E}$ such that $A \cap Q \approx E \cap Q$ and $A \cap Q' \approx E' \cap Q'$.*

This is easily proved by induction on the generation of A by $\sim$, $\cap$, $\cup$ from $\mathscr{E} \cup \{Q\} \cup \{R_{nk} \mid n, k < \omega\}$. The following lemma contains the heart of the proof of 2.2.

Lemma 2. *If $\{E_n \mid n < \omega\} \subseteq \mathscr{E}$, $A \in \mathscr{F}$ and A is the meet in $\mathscr{F}$ of $\{E_n \mid n < \omega\} \cup \{Q\}$, then there is an $E \in \mathscr{E}$ such that $E = \bigwedge_{n<\omega}^{\mathscr{E}} E_n$ and $A = E \cap Q$.*

Proof. By Lemma 1 we can choose $E \in \mathscr{E}$ so that $A \cap Q \approx E \cap Q$. But $A \subseteq Q$ so $A \approx E \cap Q$. Put $N_1 = (E \cap Q) \sim A$, $N_2 = A \sim (E \cap Q)$. Then N_1, N_2 are countable and $\in \mathscr{R}$. Now $A \supseteq (E \cap Q) \sim N_1 = E \cap (Q \sim N_1)$, and by 2.1(2) $Q \sim N_1$ intersects every basic set. Therefore $\mathrm{cl}(A) \supseteq \mathrm{cl}(E \cap (Q \sim N_1)) = E$. But $A \subseteq E_n$ for all n, hence $\mathrm{cl}(A) \subseteq \bigcap_n E_n$ (since $\bigcap_n E_n$ is closed), and so $E \subseteq \bigcap_n E_n$. Thus E is a lower bound in $\mathscr{E}$ of $\{E_n \mid n < \omega\}$. Let $E' \in \mathscr{E}$ be another lower bound. Then $E' \cap Q$ is a lower bound in $\mathscr{F}$ of $\{E_n \mid n < \omega\} \cup \{Q\}$, hence $E' \cap Q \subseteq A$. But $A \subseteq (E \cap Q) \cup N_2$, so $E' \cap Q \subseteq (E \cap Q) \cup N_2$, $(E' \sim E) \cap Q \subseteq N_2$. But by 2.1(2) Q intersects each nonempty open set in an uncountable set, whereas N_2 is countable. Therefore $E' \sim E = \varnothing$, $E' \subseteq E$. This shows that E is the greatest lower bound of $\{E_n \mid n < \omega\}$ in $\mathscr{E}$.

To complete the proof of the lemma we must show that $A = E \cap Q$, i.e., that $N_1 = N_2 = \varnothing$. Since $E \cap Q$ is a lower bound in $\mathscr{F}$ of $\{E_n \mid n < \omega\} \cup \{Q\}$, $E \cap Q \subseteq A$. Hence $N_1 = \varnothing$ and $A = (E \cap Q) \cup N_2$. Since $A \subseteq Q$ and $N_2 = A \sim (E \cap Q)$ we see that $N_2 \subseteq Q$ and N_2 is disjoint from E. Also $N_2 \subseteq A \subseteq \bigcap_n E_n$.

We shall now find the general form of elements of $\mathscr{R}$. Let S be a finite union of R_{nk}'s. Consider any $A \in \mathscr{F}$. By induction on the generation of A from

$$\mathscr{E} \cup \{Q\} \cup \{R_{nk} \mid n, k < \omega\}$$

it is directly seen (noting the disjointness of the R_{nk}'s) that $S \cap A$ is a finite union of sets of the form $R_{nk} \cap E$ $(n, k < \omega, E \in \mathscr{E})$. If $A \in \mathscr{R}$ then for some S as above, $A = S \cap A$ and therefore A has the form $\bigcup_{i<m} (R_{n_i k_i} \cap E_i)$ where $m < \omega$, and for $i < m$: $n_i, k_i < \omega$, $E_i \in \mathscr{E}$. Conversely, every set of this form is (of course) in $\mathscr{R}$.

In particular put $N_2 = \bigcup_{i<m} (R_{n_i k_i} \cap E_i)$ with m etc. as above, and suppose for contradiction that $N_2 \neq \varnothing$. Without loss of generality, $m > 0$ and $R_{n_0 k_0} \cap E_0 \neq \varnothing$. Combining this with the above properties of N_2 we get: $\varnothing \neq R_{n_0 k_0} \cap E_0 \subseteq N_2 \subseteq (\bigcap_n E_n) \cap Q \cap ({}^{\omega}2 \sim E)$. Let $k < \omega$ be larger than k_i for all $i < m$. Then $R_{n_0 k}$ is disjoint from $R_{n_i k_i}$ for all $i < m$ and hence from N_2. Also $\mathrm{cl}(R_{n_0 k_0}) = \mathrm{cl}(R_{n_0 k}) = Q_{n_0}$, hence $\mathrm{cl}(R_{n_0 k_0} \cap E_0) = \mathrm{cl}(R_{n_0 k} \cap E_0) = Q_{n_0} \cap E_0$ (because E_0 is open and closed). Now, $(\bigcap_n E_n) \cap ({}^{\omega}2 \sim E)$ is a closed set including $R_{n_0 k_0} \cap E_0$, hence it includes also $R_{n_0 k} \cap E_0$. Moreover, $R_{n_0 k_0} \cap E_0 \neq \varnothing \Rightarrow \mathrm{cl}(R_{n_0 k} \cap E_0) \neq \varnothing \Rightarrow R_{n_0 k} \cap E_0 \neq \varnothing$, and $R_{n_0 k} \subseteq Q_{n_0} \subseteq Q$. Thus, $\varnothing \neq R_{n_0 k} \cap E_0 \subseteq (\bigcap_n E_n) \cap Q \cap ({}^{\omega}2 \sim E)$. We see that $R_{n_0 k} \cap E_0$ is a lower bound in $\mathscr{F}$ of $\{E_n \mid n < \omega\} \cup \{Q\}$ which is nonempty and disjoint from E and from N_2. But $A \subseteq E \cup N_2$, so $R_{n_0 k} \cap E_0$ is disjoint from A, and $A \cup (R_{n_0 k} \cap E_0)$ is a lower bound of $\{E_n \mid n < \omega\} \cup \{Q\}$ which is strictly greater than A, in contradiction to the assumption of Lemma 2. Thus $N_2 = \emptyset$, $A = E \cap Q$ and the proof of the lemma is complete.

LEMMA 3. *Like Lemma 2 with Q replaced by Q'.*

PROOF. In the proof of Lemma 2 up to the point where it is concluded that $N_1 = \emptyset$, write everywhere "Q'" for "Q". The resulting argument is valid as it stands. We conclude that $A = (E \cap Q') \cup N_2$ where $N_2 \in \mathscr{R}$. But $A \subseteq Q'$, hence $N_2 \subseteq Q'$. On the other hand N_2 is included in a union of R_{nk}'s, so $N_2 \subseteq Q$, and thus $N_2 = \emptyset$ since $Q' = {}^{\omega}2 \sim Q$. This completes the proof

The proof of 2.2 is now easy. $\mathscr{E}$ is a subfield, hence a subalgebra, of $\mathscr{F}$. To see that it is a $< \infty$-subalgebra, it suffices to prove (remembering that $\mathscr{E}$ is countable) that if $\{E_n | n < \omega\} \subseteq \mathscr{E}$, $A \in \mathscr{F}$ and $A = \bigwedge_n^{\mathscr{F}} E_n$, then $A \in \mathscr{E}$. But if A is the meet in $\mathscr{F}$ of $\{E_n | n < \omega\}$ then $A \cap Q$ is the meet in $\mathscr{F}$ of $\{E_n | n < \omega\} \cup \{Q\}$ and $A \cap Q'$ is the meet of $\{E_n | n < \omega\} \cup \{Q'\}$. By Lemmas 2 and 3 the meet $\bigwedge_n^{\mathscr{E}} E_n$ exists (call it E) and satisfies $A \cap Q = E \cap Q$, $A \cap Q' = E \cap Q'$, and hence $A = E \in \mathscr{E}$. Q. E. D.

To prove 2.3 let $\mathscr{C}$ be the normal completion of $\mathscr{F}$. Recall that the sets $P_{mn} \in \mathscr{E}$ have been chosen so that for all n, $Q_n = \bigcap_m P_{mn}$. Put $q_n = \bigwedge_m^{\mathscr{C}} P_{mn}$ and let us "compute" q_n (since $\mathscr{F}$ is a dense subalgebra of $\mathscr{C}$, each member c of $\mathscr{C}$ can be represented by a subset of $\mathscr{F}$, namely by $\{x \in \mathscr{F} | x \leqq c\}$; computing c means finding this set):

$$\{A \in \mathscr{F} | A \leqq q_n\} = \{A \in \mathscr{F} | \forall m (A \subseteq P_{mn})\} = \{A \in \mathscr{F} | A \subseteq Q_n\}.$$

LEMMA 4. *For any $A \in \mathscr{F}$, $A \subseteq Q_n$ iff for some m, $A \subseteq \bigcup_{k<m} R_{nk}$.*

PROOF. Since $\bigcup_{k<m} R_{nk} \subseteq Q_n$ always, one direction is trivial. Now let $A \in \mathscr{F}$, $A \subseteq Q_n$. By Lemma 1 there is an $E \in \mathscr{E}$ such that $A \cap Q \approx E \cap Q$. But $A \subseteq Q$ so $A \approx E \cap Q$. As in the beginning of the proof of Lemma 2, this implies $\mathrm{cl}(A) \supseteq E$. But Q_n is closed and $\supseteq A$; hence, $Q_n \supseteq E$. Since Q_n is nowhere dense, $E = \emptyset$, so $A \approx \emptyset$. Thus A is covered by a finite union $R_{n_0 k_0} \cup \cdots \cup R_{n_i k_i} \cup \cdots$. But when $n_i \neq n$ we have $R_{n_i k_i} \cap A \subseteq Q_{n_i} \cap Q_n = \emptyset$, hence A is covered by a finite union of sets R_{nk}. Q. E. D.

It is worth noting that by Lemma 4 (or by 2.2) $Q_n \notin \mathscr{F}$ for each n.

Returning to $q_n = \bigwedge_n^{\mathscr{C}} P_{mn}$ we see that $q_n = \bigvee^{\mathscr{C}} \{A \in \mathscr{F} | A \subseteq Q_n\} = \bigvee_k^{\mathscr{C}} R_{nk}$. Hence $\bigvee_n^{\mathscr{C}} q_n = \bigvee_{n,k}^{\mathscr{C}} R_{nk}$. If we prove that $Q = \bigvee_{n,k}^{\mathscr{C}} R_{nk}$ we shall get $Q = \bigvee_n^{\mathscr{C}} q_n = \bigvee_n^{\mathscr{C}} \bigwedge_m^{\mathscr{C}} P_{mn}$, proving 2.3. Since $\mathscr{F}$ is a regular subalgebra of $\mathscr{C}$ it suffices to prove:

LEMMA 5. $Q = \bigvee^{\mathscr{F}}_{n,k} R_{nk}$.

PROOF. $Q \supseteq R_{nk}$ for all n,k. Now let $A \in \mathscr{F}$, $A \supseteq R_{nk}$ for all n,k. We shall show that $A \supseteq Q$. By Lemma 1, there is an $E \in \mathscr{E}$ such that $A \cap Q \approx E \cap Q$. Let $N_1 = (E \cap Q) \sim (A \cap Q)$ and $N_2 = (A \cap Q) \sim (E \cap Q)$, so that $N_1, N_2 \in \mathscr{R}$. Since $R_{nk} \subseteq A$ for all n,k it follows that $N_1 \subseteq A$, hence $N_1 \subseteq A \cap Q$, $N_1 = \emptyset$. Now $N_2 = (A \sim E) \cap Q$, and since N_2 is covered by a finite number of R_{nk}'s there is an m such that $N_2 \subseteq \bigcup_{i<m} Q_i$. Thus if $n \geqq m$ then for all k, $R_{nk} \cap N_2 = \emptyset$ and $R_{nk} \subseteq A \cap Q$, so $R_{nk} \subseteq E$; hence, $Q_n = \mathrm{cl}(R_{nk}) \subseteq E$, so $Q \cap ({}^{\omega}2 \sim E) \subseteq \bigcup_{i<m} Q_i$. But Q is a dense set and $\bigcup_{i<m} Q_i$ is closed, so taking the closure we get ${}^{\omega}2 \sim E \subseteq \bigcup_{i<m} Q_i$, and since $\bigcup_{i<m} Q_i$ is nowhere-dense, ${}^{\omega}2 \sim E = \emptyset$, $E = {}^{\omega}2$ and $N_2 = (A \sim E) \cap Q = \emptyset$. Thus $A \cap Q = {}^{\omega}2 \cap Q = Q$, i.e., $A \supseteq Q$. Thus Q is the least upper bound in $\mathscr{F}$ of $\{R_{nk} \mid n, k < \omega\}$. This proves Lemma 5 and completes the proof of 2.3, and hence of 1.5, 1.3.

REMARK. Define $I: \omega \to \mathscr{E}$ by $I(n) = \{x \in {}^{\omega}2 \mid x(n) = 1\}$. Take $Q, Q', Q_n (n < \omega)$ as in the example following 2.1 and define $\mathscr{F}$ accordingly ($Q = \{x \in {}^{\omega}2 \mid \exists n (\forall m \geqq n)(x(2m) = 1)\}$ etc.). The definition of Q_n and the above proof show that $Q = \|\psi\|_{\mathscr{C},I}$ where

$$\begin{aligned}\psi = {} & (p_0 \wedge p_2 \wedge p_4 \wedge p_6 \cdots) \vee (\neg p_0 \wedge p_2 \wedge p_4 \wedge p_6 \cdots) \\ & \vee (\neg p_2 \wedge p_4 \wedge p_6 \wedge p_8 \cdots) \vee (\neg p_4 \wedge p_6 \wedge p_8 \wedge p_{10} \cdots) \\ & \vee (\neg p_6 \wedge \cdots) \vee \cdots .\end{aligned}$$

Thus ψ is a simple example of a B.t. over ω which is weakly defined in $(\mathscr{F}, I)$ but not equivalent to any strongly defined B.t.

3. The case of reduced valuations

We return to the consideration of valuations over an arbitrary fixed set D. We shall use the equation $[\text{range } (I)]^{<\infty}_{\mathscr{B}} = \{\|\phi\|_{\mathscr{B},I} \mid \phi \text{ is strongly defined in } (\mathscr{B}, I)\}$, which is true in every valuation $(\mathscr{B}, I)$. If the valuation is reduced, i.e. $\mathscr{B} = [\text{range } (I)]^{<\infty}_{\mathscr{B}}$, we get that each $b \in \mathscr{B}$ has the form $\|\phi\|_{\mathscr{B},I}$ and hence there is a set T of B.t's satisfying the following:

3.1. (1) $T \supseteq \{p_i \mid i \in D\}$ and T is closed under $\neg, \wedge, \vee$;

(2) each $\phi \in T$ is strongly defined in $(\mathscr{B}, I)$;

(3) for each $b \in \mathscr{B}$ there is a $\phi \in T$ such that $b = \|\phi\|_{\mathscr{B},I}$.

From now on let $(\mathscr{B}, I)$ be a fixed reduced valuation (over D) and T a set of

B.t's as in 3.1. Let

$\Delta_1 = \{\phi \in T \mid \|\phi\|_{\mathscr{B},I} = 1\}$; $\Delta_2 = \{\bigvee X \mid X \subseteq T,\ \bigvee_{\psi \in X} \|\psi\|_{\mathscr{B},I} = 1\}$ and $\Delta = \Delta_1 \cup \Delta_2$.

Denote by $\mathscr{C}$ the normal completion of $\mathscr{B}$. We shall first show that, in a sense, Δ is a complete axiomatization of the theory of $(\mathscr{C}, I)$, and then use this result (3.2) to show that every B.t. weakly defined in $(\mathscr{B}, I)$ is equivalent to a strongly defined one.

3.2. THEOREM. *Under the assumptions and notations above, each B.t. ϕ satisfies:*

$$\Delta \vdash \phi \text{ iff } \|\phi\|_{\mathscr{C},I} = 1.$$

PROOF. Δ is a set of B.t's each of which has the value 1 in $(\mathscr{B}, I)$, hence in $(\mathscr{C}, I)$. Therefore, by the characterization of " $\vdash$ ", if $\Delta \vdash \phi$ then $\|\phi\| = 1$ (where $\|\cdot\|$ is short for $\|\cdot\|_{\mathscr{C} I}$ in this proof).

Now suppose that ϕ_0 is a B.t. such that $\Delta \nvdash \phi_0$ and let $(\mathscr{C}', I')$ be a valuation such that $\mathscr{C}'$ is complete, $\|\chi\|_{\mathscr{C}',I'} = 1$ for all $\chi \in \Delta$, and $\|\phi_0\|_{\mathscr{C}',I'} \neq 1$.

We assert that there is a complete homomorphism $j: \mathscr{B} \to \mathscr{C}'$ given by $j(\|\phi\|_{\mathscr{B},I}) = \|\phi\|_{\mathscr{C}',I'}$ $(\phi \in T)$, or in short, $j(\|\phi\|) = \|\phi\|'$ for $\phi \in T$.

To see that this equation defines a single-valued function note that if $\phi, \psi \in T$ and $\|\phi\| = \|\psi\|$ then $(\phi \leftrightarrow \psi) \in \Delta_1 \subseteq \Delta$ (because T is closed under $\neg, \wedge, \vee$, hence under $\leftrightarrow$), and hence $\|\phi \leftrightarrow \psi\|' = 1$, $\|\phi\|' = \|\psi\|'$. Thus $\|\phi\| = \|\psi\| \Rightarrow \|\phi\|' = \|\psi\|'$ (for $\phi, \psi \in T$), and j is single-valued.

By 3.1(3) $\mathrm{dom}(j) = \mathscr{B}$. Since T is closed under $\neg, \wedge, \vee$, j is a homomorphism from $\mathscr{B}$ into $\mathscr{C}'$. To prove that j is complete it suffices to show that if $A \subseteq \mathscr{B}$, $\bigvee^{\mathscr{B}} A = 1$ then $\bigvee^{\mathscr{C}'}_{a \in A} j(a) = 1$. But letting $X = \{\phi \in T \mid \|\phi\| \in A\}$ we have (by 3.1(3)) $A = \{\|\phi\| \mid \phi \in X\}$ and so, if $\bigvee^{\mathscr{B}} A = 1$ then $(\bigvee X) \in \Delta_2 \subseteq \Delta$, hence $\|\bigvee X\|' = 1$. But $\|\bigvee X\|' = \bigvee^{\mathscr{C}'}_{\phi \in X} \|\phi\|' = \bigvee^{\mathscr{C}'}_{a \in A} j(a)$, so $\bigvee^{\mathscr{C}'}_{a \in A} j(a) = 1$. Thus j is complete, and we can use 1.2 to extend it to a complete homomorphism $J: \mathscr{C} \to \mathscr{C}'$. For any $i \in D$ we have $J(I(i)) = J(\|p_i\|) = j(\|p_i\|) = \|p_i\|' = I'(i)$, hence $I' = J \circ I$. By 1.1 we conclude that $\|\phi\|' = J(\|\phi\|)$ for every B.t. ϕ. Now, $(\mathscr{C}, I')$ has been chosen such that $\|\phi_0\|' \neq 1$. Hence $\|\phi_0\| \neq 1$.

We have thus shown that for any B.t. ϕ_0, $\Delta \nvdash \phi_0 \Rightarrow \|\phi_0\|_{\mathscr{B},I} \neq 1$, completing the proof of 3.2.

We are now ready to discuss weakly defined B.t's. Suppose ϕ is weakly defined in $(\mathscr{B}, I)$ and choose (by 3.1(3)) some $\psi \in T$ so that $\|\phi\|_{\mathscr{C},I} = \|\psi\|_{\mathscr{B} I}$. Then

$\|\phi \leftrightarrow \psi\|_{\mathscr{C},I} = 1$, hence by 3.2, $\Delta \vdash \phi \leftrightarrow \psi$, which is equivalent to $\vdash (\bigwedge\Delta) \rightarrow (\phi \leftrightarrow \psi)$. Denoting $\sigma = \neg \bigwedge \Delta$ we get: $\vdash \neg \sigma \rightarrow (\phi \leftrightarrow \psi)$, hence $\vdash \phi \leftrightarrow (\sigma \wedge \phi) \vee (\neg \sigma \wedge \psi)$, i.e., $\phi \equiv (\sigma \wedge \phi) \vee (\neg \sigma \wedge \psi)$.

By the definition of Δ, $\bigwedge\Delta$ is strongly defined in $(\mathscr{B},I)$ and has the value 1. Therefore σ and $\neg \sigma$ are strongly defined and so is ψ (because $\psi \in T$). Note also that $\|\sigma\|_{\mathscr{B},I} = 0$. If we can prove that $\sigma \wedge \phi$ is equivalent to some B.t. τ strongly defined in $(\mathscr{B},I)$, we shall get $\phi \equiv \tau \vee (\neg \sigma \wedge \psi)$, and the B.t. $\tau \vee (\neg \sigma \wedge \psi)$ is strongly defined. Therefore, the proof of (*) of §1 for the reduced valuation $(\mathscr{B},I)$ will be complete if we prove the following lemma.

LEMMA. *Let σ be a B.t. strongly defined and having value* 0 *in $(\mathscr{B},I)$. Then for each B.t. ϕ there is a B.t. τ strongly defined in $(\mathscr{B},I)$ such that $\sigma \wedge \phi \equiv \tau$.*

PROOF. By induction on ϕ. If ϕ is atomic take $\tau = \sigma \wedge \phi$. Next suppose $\phi = \neg \phi_1$. By the induction hypothesis, there is some good $\tau_1 \equiv \sigma \wedge \phi_1$ ("good" means strongly defined in $(\mathscr{B},I)$). Take $\tau = \sigma \wedge \neg \tau_1$. Then τ is good and $\tau \equiv \sigma \wedge \neg(\sigma \wedge \phi_1) \equiv \sigma \wedge \phi$.

Now consider the case $\phi = \bigvee X$. By the induction hypothesis, find for each $\psi \in X$ a good $\tau_\psi \equiv \sigma \wedge \psi$. Then $\sigma \wedge \phi \equiv \bigvee_{\psi \in X}(\sigma \wedge \psi) \equiv \bigvee_{\psi \in X}\tau_\psi$, and take $\tau = \bigvee_{\psi \in X}\tau_\psi$. Since for each ψ τ_ψ is good and $\tau_\psi \wedge \sigma \equiv \tau_\psi$, we conclude that for each ψ $\|\tau_\psi\|_{\mathscr{B},I} = 0$, so τ is good too and $\tau \equiv \sigma \wedge \phi$.

If $\phi = \bigwedge X$ then $\phi \equiv \neg \bigvee_{\psi \in X} \neg \psi$, and we can find τ by going back to the previous cases (or directly). This completes the induction, and hence the proof that every reduced valuation satisfies (*) of §1.

4. Proof of 1.4

Consider a valuation $(\mathscr{B},I)$ and denote $\mathscr{B}_0 = [\text{range (I)}]^{<\infty}_{\mathscr{B}}$, $\mathscr{C}$ = normal completion of $\mathscr{B}$, $\mathscr{C}_0$ = normal completion of $\mathscr{B}_0$. Suppose that $(\mathscr{B},I)$ is a regular valuation. Then the inclusion embedding of $\mathscr{B}_0$ in $\mathscr{B}$ is complete, and by 1.2 it can be extended to a complete embedding of $\mathscr{C}_0$ in $\mathscr{C}$. We can identify $\mathscr{C}_0$ with its image under this embedding and so assume that $\mathscr{C}_0$ is a regular subalgebra of $\mathscr{C}$ in which $\mathscr{B}_0$ is dense.

Thus we have

$$\begin{array}{ccccc} & & \mathscr{B} & & \\ & \nearrow & & \searrow & \\ \mathscr{B}_0 & & & & \mathscr{C} \\ & \searrow & & \nearrow & \\ & & \mathscr{C}_0 & & \end{array}$$

where each arrow is a complete inclusion-embedding.

It is easy to see (without using regularity) that $[\text{range (I)}]_{\mathscr{B}_0}^{<\infty} = \mathscr{B}_0$, so that $(\mathscr{B}_0, I)$ is a reduced valuation.

Let ϕ be a B.t. weakly defined in $(\mathscr{B}, I)$. Thus $\| \phi \|_{\mathscr{C}, I} \in \mathscr{B}$. But $\mathscr{C}_0$ is a regular subalgebra of $\mathscr{C}$ and is complete so $\| \phi \|_{\mathscr{C}_0, I} \in \mathscr{C}_0$ and $\| \phi \|_{\mathscr{C}_0, I} = \| \phi \|_{\mathscr{C}_0, I}$

LEMMA. $\mathscr{B} \cap \mathscr{C}_0 = \mathscr{B}_0$.

PROOF. We need only prove that $b \in \mathscr{B} \cap \mathscr{C}_0 \Rightarrow b \in \mathscr{B}_0$. Let $b \in \mathscr{B} \cap \mathscr{C}_0$. Since $\mathscr{B}_0$ is dense in $\mathscr{C}_0$ there is an $A \subseteq \mathscr{B}_0$ such that $b = \bigvee^{\mathscr{C}_0} A = \bigvee^{\mathscr{C}} A = \bigvee^{\mathscr{B}} A$. But $\mathscr{B}_0$ is a $< \infty$-subalgebra of $\mathscr{B}$ so $b \in \mathscr{B}_0$.

Returning to the weakly defined B.t. ϕ we see that $\| \phi \|_{\mathscr{C}.I} = \| \phi \|_{\mathscr{C}_0.I} \in \mathscr{B} \cap \mathscr{C}_0 = \mathscr{B}_0$, so ϕ is weakly defined also in the reduced valuation $(\mathscr{B}_0, I)$. By §3 there is a B.t. ψ strongly defined in $(\mathscr{B}_0, I)$ such that $\phi \equiv \psi$. Since $\mathscr{B}_0$ is a regular subalgebra of $\mathscr{B}$, ψ is strongly defined also in $(\mathscr{B}, I)$, by 1.1. This proves that $(\mathscr{B}, I)$ satisfies (*) of §1.

5. Concluding remarks

Let $(\mathscr{B}, I)$ be a valuation, and denote $\mathscr{B}_0 = [\text{range}(I)]_{\mathscr{B}}^{<\infty}$, $\mathscr{C}$ = normal completion of $\mathscr{B}$ and $\mathscr{C}_0 = [\mathscr{B}_0]_{\mathscr{C}}^{<\infty} = [\text{range (I)}]_{\mathscr{C}}^{<\infty}$. It is easy to see that $\mathscr{C}_0$ is a normal completion of $\mathscr{B}_0$ iff $(\mathscr{B}, I)$ is regular (for one direction, see §4). Our counterexample in §2 worked because in that case $\mathscr{B} \cap \mathscr{C}_0 \supsetneq \mathscr{B}_0$ (in the notation of §2, $\mathscr{B} = \mathscr{F}$, $\mathscr{B}_0 = [\{P_{mn} \mid m, n < \omega\}]_{\mathscr{F}}^{<\infty} \subseteq \mathscr{E}$, and $Q \in \mathscr{C}_0$ because $Q = \bigvee_n^{\mathscr{C}} \bigwedge_m^{\mathscr{C}} P_{mn}$). Generally, when $\mathscr{B} \cap \mathscr{C}_0 \supsetneq \mathscr{B}_0$ one can find a B.t. ψ such that $\| \psi \|_{\mathscr{C}.I} \in \mathscr{B} \cap \mathscr{C}_0 \sim \mathscr{B}_0$, and so ψ is weakly defined in $(\mathscr{B}, I)$, but every strongly defined ϕ satisfies $\| \phi \|_{\mathscr{C}.I} = \| \phi \|_{\mathscr{B}.I} \in \mathscr{B}_0$ and so $\phi \not\equiv \psi$.

The following questions naturally present themselves:

1) Find a simpler example of a valuation $(\mathscr{B}, I)$ (over a countably infinite set) with the property that $\mathscr{B} \cap \mathscr{C}_0 \neq \mathscr{B}_0$. It is not excluded that $\mathscr{B}$ have the isomorphism type of $\mathscr{F}$ of §2, but the description and the proofs of the properties may perhaps be simplified. (Note that it is trivial to find a field of subsets of a countable set isomorphic to $\mathscr{F}$: let $X \subseteq {}^{\omega}2$ be a countable set that intersects every nonempty member of $\mathscr{F}$, and let $\mathscr{F}' = \{A \cap X \mid A \in \mathscr{F}\}$.)

2) Find an example, or prove there is none, of a valuation $(\mathscr{B}, I)$ such that $\mathscr{B} \cap \mathscr{C}_0 = \mathscr{B}_0$ and yet there is a B.t. ψ weakly defined in $(\mathscr{B}, I)$ which is not equivalent to any strongly defined one.

3) Find a necessary and sufficient condition for a valuation $(\mathscr{B}, I)$ to satisfy (*) of §1.

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